Combinatorial Networks
Week 1, March 11-12

1 Notes on March 11

1.1 The Pigeonhole Principle

The Pigeonhole Principle If n objects are placed in k holes, where $n > k$, there exists a box with more than one objects.

1.1. Theorem Given a simple graph on n vertices, there are two vertices of the same degree.
Proof: Let G be a simple graph on n vertices. There no harm in assuming that G has no vertex of degree 0, then the degree of any vertex must be in $\{1, ..., n-1\}$. Thus by P-P.(The Pigeonhole Principle), there must be two vertices of the same degree.

Examples

1.2. Subsets with divisions Let $[2n] = \{1, 2, ..., 2n\}$. Consider all the subsets $S \subset [2n]$, such that no distinct $i, j \in S$ satisfying $i \mid j$. What is the maximal number of $\mid S \mid$?
It's obvious that $T = \{n+1, ..., n+n\}$ with n elements satisfies the desired condition. We claim that this maximal number is n indeed. Assume that there is a subset S of $[2n]$ satisfying the condition but $\mid S \mid \geq n+1$. For any odd $a \in [2n]$, let $C_a = \{a \cdot 2^k, k \geq 0\} \cap [2n]$. Since $[2n] = \cup_{a \in [2n], odd} C_a$ and there are n such sets totally, by P-P., there exist distinct $i, j \in S \cap C_a$ for some odd $a \in [2n]$. Hence either $i \mid j$ or $j \mid i$, contradicts with the definition of S.

1.3. Theorem For any $x \in \mathbb{R}$ and integer $n > 0$, there is a rational number p/q, such that $\mid x - p/q \mid < 1/nq$.
Proof: Exercise.

1.4. Theorem(Erdös-Szekers) For any sequence of length $mn+1$ distinct real numbers $a_0, a_1, ...a_{mn}$, there is an increasing subsequence of length $m+1$ or a decreasing subsequence of length $n+1$.
Proof: For any $0 \leq i \leq mn$, let t_i denote the maximum length of an increasing subsequence starting with a_i. Assume all $t_i \in \{1, 2, ..., m\}$. For $1 \leq j \leq m$, define $S_j = \{i : t_i = j\}$. By P-P., there is j_0 such that $\mid S_{j_0} \mid \geq n+1$.
Let $i_0 < i_1 < ... < i_n \in S_{j_0}$. We claim that $a_{i_0} \geq a_{i_1} \geq ... \geq a_{i_n}$, so it's a decreasing subsequence of length $n+1$. If it doesn't hold, say, $a_{i_0} > a_{i_1}$, by using the increasing subsequence of length j_0 starting with a_{i_1}, we get an increasing subsequence of length j_0+1 starting with a_{i_0}. Contradiction!

Exercise Find a sequence of length mn such that there is no increasing subsequence of length $m+1$ nor decreasing subsequence of length $n+1$.

1.2 Double Counting

1.5. Lemma For any simple graph G, $\sum_{v \in V} d(v) = 2 \mid E(G) \mid$.
Proof: For any vertex v and any edge e, define $i(v,e) = 1$ if $v \sim e$, else $i(v,e) = 0$. Note that

$$\sum_{v \in V} \sum_{e \in E} i(v,e) = \sum_{e \in E} \sum_{v \in V} i(v,e)$$

.
It implies

$$\sum_{v \in V} d(v) = \sum_{e \in E} 2 = 2 \mid E(G) \mid$$

.

□

1.6. Theorem Let $t(n)$ be the number of divisors of n and

$$\bar{t}(n) = \frac{1}{n} \sum_{j=1}^{n} t(j),$$

Then $\bar{t}(n) \sim H(n)$, as $n \to \infty$, where

$$H(n) = \sum_{i=1}^{n} \frac{1}{i}.$$

Proof: For any integer i, j, if $i \mid j$, define $d(i,j) = 1$, else define $d(i,j) = 0$. Then we have

$$t(j) = \sum_{i=1}^{j} d(i,j).$$

Since

$$\sum_{j=1}^{n} \sum_{i=1}^{n} d(i,j) = \sum_{i=1}^{n} \sum_{j=1}^{n} d(i,j),$$

we have

$$\sum_{j=1}^{n} t(j) = \sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor = n \sum_{i=1}^{n} \frac{1}{i} + O(n),$$

as $n \to \infty$.
It implies

$$\bar{t}(n) = \frac{1}{n} \sum_{j=1}^{n} t(j) = H(n) + O(1) \sim H(n).$$

□

1.3 Binomial Theorem

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

Definition: For integer k, and polynomial $f(x)$, let $[x^k]f$ be the coefficient in the term in $f(x)$.

1.7. Fact For any polynomials $f_1(x), ..., f_k(x)$, let

$$f(x) = \prod_{i=1}^{k} f_i(x),$$

then for any integer n,

$$[x^n]f = \sum_{i_1+...+i_k=n} \prod_{j=1}^{k} [x^{i_j}]f_j.$$

1.8. Fact

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2.$$

Proof:

$$\begin{aligned}\binom{2n}{n} = [x^n](1+x)^{2n} &= \sum_{i+j=n} ([x^i](1+x)^n)([x^j](1+x)^n) \\ &= \sum_{i+j=n} \binom{n}{i}\binom{n}{j} \\ &= \sum_{i=0}^{n} \binom{n}{i}^2\end{aligned}$$

□

1.9. Fact For all positive integer n,

$$e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n.$$

Proof: Since

$$\int_{k-1}^{k} \ln x \; dx \leq \ln k \leq \int_{k}^{k+1} \ln x \; dx,$$

we get

$$\int_1^n \ln x \; dx \le \ln n! \le \int_2^{n+1} \ln x \; dx.$$

That is,

$$\ln\left(\frac{n}{e}\right)^n + 1 \le \ln n! \le \ln\left(\frac{n+1}{e}\right)^{n+1} - \ln\left(\frac{2}{e}\right)^2,$$

which implies

$$e\left(\frac{n}{e}\right)^n \le n! \le \left(\frac{n+1}{e}\right)^{n+1}\left(\frac{e}{2}\right)^2.$$

Note that

$$\left(\frac{n+1}{e}\right)^{n+1} = \left(\frac{n}{e}\right)^{n+1}\left(1+\frac{1}{n}\right)^{n+1} \le 4\left(\frac{n}{e}\right)^{n+1}.$$

Thus,

$$e\left(\frac{n}{e}\right)^n \le n! \le en\left(\frac{n}{e}\right)^n$$

□

Remark: Stirling's Formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \; as \; n \to \infty.$$

1.10. Theorem For $1 \le k \le n$,

$$\sum_{i=0}^{k} \binom{n}{k} \le \left(\frac{en}{k}\right)^k.$$

Proof: For any $x \in (0, 1]$, note that

$$x^{-k} \sum_{i=0}^{k} \binom{n}{k} x^i \le x^{-k}(1+x)^n$$

and

$$1 + x \le e^x.$$

From them we can get

$$\sum_{i=0}^{k} \binom{n}{k} \le \sum_{i=0}^{k} \binom{n}{k} x^{i-k} \le \frac{(1+x)^n}{x^k} \le \frac{e^{nx}}{x^k}.$$

At last, let $x = k/n$, and substitute it into the formula above, then we get

$$\sum_{i=0}^{k} \binom{n}{k} \leq \left(\frac{en}{k}\right)^k,$$

as desired.

□

1.11. Corollary

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

2 Notes on March 12

2.1. Binomial Theorem For any integer $n > 0$, and any real x,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

2.2. Newton's Binomial Theorem For any real r, and any real $x \in (-1, 1)$,

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k.$$

Here,

$$\binom{r}{k} = \frac{r(r-1)...(r-k+1)}{k!}.$$

2.3. Corollary Let $x \in (-1, 1)$, and $r = -n$, where integer $n > 0$,

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k.$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k.$$

2.1 Inclusion-Exclusion

2.4. Theorem(Inclusion-Exclusion) For subsets $A_1, ..., A_n \subset X$,

$$| X \setminus \bigcup_{i=1}^{n} A_i | = \sum_{I \subset [n]} (-1)^{|I|} | \bigcap_{i \in I} A_i | .$$

Proof: For any subset $A \subset X$, define its characteristic function $f_A(x)$, where $f_A(x) = 1$ if $x \in A$, else $f_A(x) = 0$, then

$$\sum_{x \in X} f_A(x) = \mid A \mid .$$

Consider

$$F(x) \triangleq \prod_{i=1}^{n} (1 - f_{A_i}(x)) = \sum_{I \subset [n]} (-1)^{|I|} \prod_{i \in I} f_{A_i}(x).$$

Note that $\prod_{i \in I} f_{A_i}(x)$ is the characteristic function of $\bigcap_{i \in I} A_i$, and that $F(x)$ is the characteristic function of $X \setminus \bigcup_{i=1}^{n} A_i$, since $F(x) = 1$ if and only if $x \notin A_i$ for all $i = 1, 2, ..., n$, and else $F(x) = 0$. So by what have observed before,

$$\begin{aligned} \mid X \setminus \bigcup_{i=1}^{n} A_i \mid &= \sum_{x \in X} F(x) \\ &= \sum_{x \in X} \sum_{I \subset [n]} (-1)^{|I|} \prod_{i \in I} f_{A_i}(x) \\ &= \sum_{I \subset [n]} (-1)^{|I|} \sum_{x \in X} \prod_{i \in I} f_{A_i}(x) \\ &= \sum_{I \subset [n]} (-1)^{|I|} \mid \bigcap_{i \in I} A_i \mid \end{aligned}$$

□

2.5. Corollary

$$\mid \bigcup_{i=1}^{n} A_i \mid = \mid X \mid - \mid X \setminus \bigcup_{i=1}^{n} \mid = \sum_{\emptyset \neq I \subset [n]} (-1)^{|I|+1} \mid \bigcap_{i \in I} A_i \mid .$$

Definition: A derangement $\pi : [n] \to [n]$ is a bijection(permutation) such that $\pi(i) \neq i$ for all $i \in [n]$.

2.6. Theorem Let D_n be the set of all derangement from $[n]$ to $[n]$, then

$$\mid D_n \mid = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

Proof: Let X be the set of all the bijections from $[n]$ to $[n]$, and for each $i \in [n]$, let A_i be the set $\{\pi \in X : \pi(i) = i\}$.

Since $D_n = X \setminus \bigcup_{i=1}^n A_i$, and for each $I \in [n]$, $\mid \bigcap_{i\in I} A_i \mid= (n- \mid I \mid)!$, then by Inclusion-Exclusion, we get

$$\mid D_n \mid= \sum_{I\subset[n]} (-1)^{|I|} \mid \bigcap_{i\in I} A_i \mid= \sum_{k=0}^{n}(-1)^k \binom{n}{k}(n-k)! = n!\sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

□

2.7. Corollary

$$\mid D_n \mid\sim \frac{n!}{e},\ as\ n \to \infty.$$

Exercise Let $\varphi(n)$ be the number of integers $m \in [n]$ relatively prime to n. If $n = p_1^{a_1}...p_t^{a_t}$, where $a_1, ..., a_t$ are positive integers and $p_1, ..., p_t$ are different primes, then

$$\varphi(n) = n\prod_{i=1}^{t}\left(1-\frac{1}{p_i}\right).$$

2.8. Theorem Suppose that m, n are positive integers with $m \geq n$, then the number of surjections from $[m]$ to $[n]$ is

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}(n-k)^m.$$

Proof: Let $X = \{f : [m] \to [n]\}$, and $A_i = \{f : [m] \to [n] \setminus \{i\}\}$ for each $i \in [n]$. Then

$$X \setminus \bigcup_{i=1}^{n} A_i = \{all\ surjections\ from\ [m]\ to\ [n]\}\ .$$

By Inclusion-Exclusion,

$$\mid X \setminus \bigcup_{i=1}^{n} A_i \mid= \sum_{I\subset[n]} (-1)^{|I|} \mid \bigcap_{i\in I} A_i \mid= \sum_{k=0}^{n}(-1)^k \binom{n}{k}(n-k)^m.$$

□

2.2 Generating Function

Definition: Given $\{a_n\}_{n\geq 0}$, $f(x) \triangleq \sum_{n\geq 0} a_n x^n$ is called the generating function of $\{a_n\}_{n\geq 0}$.
Addtion: $f(x) + g(x) = \sum_{n\geq 0}(a_n + b_n)x^n$;
Multiplying: $f(x)g(x) = \sum_{n\geq 0} c_n x^n$, where $c_n = \sum_{i+j=n} a_i b_j$.

Definition: A triangulation of n-gon, is that to join the vertices to divide this n-gon into triangles with intersecting only at vertices. Let b_{n-1} is the number of triangulations of n-gon where $n \geq 3$ and $b_1 \triangleq 1$, $b_0 \triangleq 0$.
These numbers $b_0, b_1, b_2, ...,$ are called Catalan numbers.

2.9. Theorem For all $n \geq 1$,

$$b_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Proof: By the definition of triangulation and b_{n-1}, where $n \geq 3$,

$$b_{n-1} = \sum_{i=3}^{n} b_{i-2}b_{n-i+1} \ .$$

Since $b_1 = 1$ and $b_0 = 0$, it implies for $k \geq 2$,

$$b_k = \sum_{i=0}^{k} b_i b_{k-i} \ .$$

Let $f(x)$ be the generating function of $\{b_k\}_{k\geq 0}$, that is,

$$f(x) = \sum_{k=0}^{\infty} b_k x^k = x + \sum_{k=2}^{\infty} b_k x^k = x + \sum_{k=0}^{\infty}\sum_{i=0}^{k} b_i b_{k-i} x^k = x + f(x)f(x).$$

Thus $f^2(x) - f(x) + x = 0$, and since $b_0 = f(0) = 0$, which implies

$$f(x) = \frac{1-\sqrt{1-4x}}{2} \ .$$

By Newton's Binomial Theorem,

$$f(x) = \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}(-4)^k x^k = \sum_{k=1}^{\infty}\frac{(2k-2)!}{k!(k-1)!}x^k.$$

Hence

$$b_k = \frac{(2k-2)!}{k!(k-1)!} = \frac{1}{k}\binom{2k-2}{k-1}.$$

□

Exercise Let p be a positive integer, prove that
if p is odd,

$$|\bigcup_{i=0}^{n} A_i| \leq \sum_{I\subset[n], 1\leq|I|\leq p} (-1)^{|I|+1} |\bigcap_{i\in I} A_i|;$$

if p is even,

$$|\bigcup_{i=0}^{n} A_i| \geq \sum_{I\subset[n], 1\leq|I|\leq p} (-1)^{|I|+1} |\bigcap_{i\in I} A_i| \ .$$